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An Accelerated Flow around a Sphere

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Abstract—A solution for the initial laminar boundary-layer flow around a sphere which suddenly starts to move with the velocity $U = U_0 + U_1 t + U_2 t^2$ has been developed in powers of the time from the start of the motion. The time series is valid past the time when separation occurs and a number of characteristic flow properties can be calculated to third order.

Keywords—Boundary-layer, Unsteady, Incompressible, Viscous, Sphere.

INTRODUCTION

In this paper, we study unsteady boundary-layer flow generated by a sphere of radius a: at time $t = 0$, the viscous incompressible fluid surrounding a stationary sphere suddenly starts to move with the velocity $U = U_0 + U_1 t + U_2 t^2$ along an axis of symmetry. There are three basic parameters in the problem. One is the Reynolds number $R = 2aU_0/\nu$, where U_0 is the initial flow velocity, which is assumed to be nonzero, and ν is the coefficient of the kinematic viscosity of the fluid. The others are the parameters $\alpha_1 = aU_1/U_0^2$ and $\alpha_2 = a^2U_2/U_0^3$.

The uniform transient flow over a sphere ($U_1 = U_2 = 0$) has previously been studied using boundary-layer theory [1–3]. It seems one of the basic difficulties in the direct use of numerical methods is the specification of the initial conditions corresponding to the impulsive start. The numerical method of Rimon and Cheng cannot describe the initial solution correctly, and they give few results which describe the flow at early times and concentrate mainly on the steady-state results. In fact, it is reported by Rimon and Cheng that the most troublesome flow to calculate is the initial flow.

The object of the present paper is to give some of the details of the expansion of the solution in powers of the time when the external flow has the given form $U = U_0 + U_1 t + U_2 t^2$. The essential point here is that this expansion gives the exact solution for small times which may be used to check the initial flow details in numerous cases of interest, especially at high Reynolds numbers. For example, by making $U_1 = 0$ and appropriate choice of U_2, we can obtain the initial motion for the case of an oscillating flow in which $U = U_0 \cos(\alpha t) \sim U_0(1 - \alpha^2 t^2/2)$. We shall not, however, consider the oscillating case in detail here. The most recent theoretical study of initial oscillatory flow past a sphere for boundary-layer case has been made by Kocabiyik [4]. Her study dealt with the case when the flow oscillations are represented by the velocity $U\cos(\beta t)$ so that $\beta = O(1)$. It is noted that the initial structure of this flow would be different if it were started using $\sin(\beta t)$ due to the difference between the small time series expansions.

These calculations were checked with the symbolic computation program Maple V Release 2.

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The present analysis adopts basically the same type of perturbation method as that used by Kocabiyik [4]. Modified polar coordinates (ξ, θ) in a plane through the axis of the sphere are used, where $\xi = \log(r/a)$, with the origin at the center of the sphere. The axis of the motion coincides with $\theta = 0$. The problem of finding the initial flow is overcome by introducing boundary-layer variables into the equations for the stream-function and vorticity. The initial singularity in the vorticity is removed by a transformation. The vorticity and stream-function are then expanded in series of Legendre functions with the argument $z = \cos\theta$. This reduces the equations of motion to a set of time dependent differential equations in the radial variable. These are truncated and solved analytically.

FORMULATION OF THE PROBLEM AND ANALYSIS

The equation of continuity is satisfied by introducing the dimensionless stream function $\psi(\xi, \theta)$ defined by the equations

$$u = \frac{e^{-2\xi}}{\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad v = -\frac{e^{-2\xi}}{\sin\theta}\frac{\partial\psi}{\partial\xi}, \tag{1}$$

where (u, v) are the dimensionless radial and transverse components of velocity obtained by dividing the corresponding dimensional components by the initial flow velocity U_0. The other dependent variable to be used is the dimensionless vorticity $\zeta(\xi, \theta)$ defined by the equation

$$\zeta = e^{-\xi}\left(\frac{\partial v}{\partial\xi} + v - \frac{\partial u}{\partial\theta}\right). \tag{2}$$

The equations satisfied by ψ and ζ are then

$$\begin{aligned} \frac{\partial\zeta}{\partial\tau} = {} & \frac{2e^{-2\xi}}{R}\left[\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial\zeta}{\partial\xi} + \cot\theta\frac{\partial\zeta}{\partial\theta} + \frac{\partial^2\zeta}{\partial\theta^2} - \frac{\zeta}{\sin^2\theta}\right] \\ & + \frac{e^{-3\xi}}{\sin\theta}\left[\left(\frac{\partial\psi}{\partial\xi}\frac{\partial\zeta}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial\xi}\right) + \left(\frac{\partial\psi}{\partial\theta} - \cot\theta\frac{\partial\psi}{\partial\xi}\right)\zeta\right], \end{aligned} \tag{3}$$

$$\frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial\psi}{\partial\xi} - \cot\theta\frac{\partial\psi}{\partial\theta} + \frac{\partial^2\psi}{\partial\theta^2} + e^{3\xi}\sin\theta\zeta = 0, \tag{4}$$

where $\tau = U_0 t/a$, $R = 2aU_0/\nu$ is the Reynolds number, ν being the coefficient of kinematic viscosity. Equations (3) and (4) are those considered by Kocabiyik [4] in the case of oscillatory flow around a sphere. In the present case, acceleration of the fluid enters through the parameters $\alpha_1 = aU_1/U_0^2$ and $\alpha_2 = a^2U_2/U_0^3$ in the boundary conditions, which may be stated as

$$\psi = \frac{\partial\psi}{\partial\xi} = 0 \quad \text{when } \xi = 0; \qquad \psi \sim \frac{1}{2}e^{2\xi}\left(1 + \alpha_1\tau + \alpha_2\tau^2\right)\sin^2\theta \quad \text{as } \xi \to \infty; \tag{5}$$

$$\psi(\xi, \theta, \tau) = \zeta(\xi, \theta, \tau) = 0 \qquad \text{when } \theta = 0, \quad \theta = \pi. \tag{6}$$

In the present analysis, the calculations are carried out on the basis of the method of solution adopted by Kocabiyik [4] in which the functions ψ and ζ were expressed in the form of the series

$$\psi(\xi, \theta, \tau) = e^{(1/2)\xi}\sum_{n=1}^{\infty} f_n(\xi, \tau)\int_z^1 P_n(s)\,ds, \qquad \zeta(\xi, \theta, \tau) = \sum_{n=1}^{\infty} g_n(\xi, \tau)P_n^{(1)}(z), \tag{7}$$

to determine the initial flow in the boundary-layer mainly by analytical methods for small values of τ. Here, $P_n(z)$ and $P_n^{(1)}(z)$ are, respectively, the Legendre and first associated Legendre polynomials of order n, and $z = \cos\theta$. These functions form a complete orthogonal set in the range $z = -1$ to $z = 1$ and the terms in the expansions reflect the correct symmetry properties of the corresponding functions over this range. The boundary conditions become

$$f_n = \frac{\partial f_n}{\partial\xi} = 0 \quad \text{when } \xi = 0; \qquad f_n \sim e^{(3/2)\xi}\left(1 + \alpha_1\tau + \alpha_2\tau^2\right)\delta_{n,1}, \quad \text{as } \xi \to \infty, \tag{8}$$

where $\delta_{1,1} = 1$ and $\delta_{n,1} = 0$ if $n \neq 1$. It may be shown, following Dennis and Walker [3], that (8) can be used to deduce from (4) a further set of conditions of global type, namely

$$\int_0^\infty \mathrm{e}^{-(n-2)\xi} g_n(\xi, \tau)\, d\xi = \frac{3}{2}\left(1 + \alpha_1\tau + \alpha_2\tau^2\right)\delta_{n,1}. \tag{9}$$

In order to deal with the initial flow we make the transformations

$$\xi = kx, \quad k = 2\left(\frac{2\tau}{R}\right)^{1/2}; \qquad \psi = k\Psi, \quad \zeta = \frac{\omega}{k}. \tag{10}$$

In the high-Reynolds number limit, equations (1) and (2), after the use of boundary-layer scaling (10), may be written as

$$\frac{\partial^2\omega}{\partial x^2} + 2x\frac{\partial\omega}{\partial x} + 2\omega = 4\tau\left[\frac{\partial\omega}{\partial\tau} - \frac{1}{\sin\theta}\left(\frac{\partial\Psi}{\partial x}\frac{\partial\omega}{\partial\theta} - \frac{\partial\Psi}{\partial\theta}\frac{\partial\omega}{\partial x} - \cot\theta\frac{\partial\Psi}{\partial x}\omega\right)\right], \tag{11}$$

$$\frac{\partial^2\Psi}{\partial x^2} + \sin\theta\,\omega = 0. \tag{12}$$

These are the final equations which are solved. The boundary conditions utilized in conjunction with (11) and (12) are the transformed conditions (5), (9) after (11) and (12) have been applied. This gives

$$\Psi = \frac{\partial\Psi}{\partial\xi} = 0 \qquad \text{when } \xi = 0, \tag{13}$$

$$\int_0^\infty \int_{-1}^1 \mathrm{e}^{-nkx}\omega(x,\theta,\tau)P_n^{(1)}(z)\,dz\,dx = \frac{3n(n+1)}{2n+1}\left(1 + \alpha_1\tau + \alpha_2\tau^2\right)\delta_{n,1}. \tag{14}$$

We now expand the functions Ψ and ω in series of powers of τ with functional coefficients consisting of functions of x with Legendre functions of argument $z = \cos\theta$ in the form

$$\Psi(x,\theta,\tau) = \sum_{n=0}^{\infty}\Psi_n(x,\theta)\tau^n, \qquad \omega(x,\theta,\tau) = \sum_{n=0}^{\infty}\omega_n(x,\theta)\tau^n. \tag{15}$$

The differential equations for the functions $\Psi_n(x,\theta)$ and $\omega_n(x,\theta)$ and the boundary conditions satisfied by these functions can be easily found. The leading terms ω_0 and ψ_0 of the expansions are the same initial solutions that are given by Kocabiyik [4], namely

$$w_0 = -3\pi^{-1/2}\,\mathrm{e}^{-x^2}\sin\theta, \qquad \Psi_0 = \frac{3}{2}\left[x\,\mathrm{erf}(x) - \pi^{-1/2}\left(1 - \mathrm{e}^{-x^2}\right)\right]\sin^2\theta. \tag{16}$$

From these it is found that ω_1 satisfies the equation

$$\frac{\partial^2\omega_1}{\partial x^2} + 2x\frac{\partial\omega_1}{\partial x} - 2\omega_1 = 36\,\pi^{-1}\left[\pi^{1/2}x\,\mathrm{erf}(x) - \left(1 - \mathrm{e}^{-x^2}\right)\right]\sin 2\theta. \tag{17}$$

The solution satisfying all the conditions is

$$\begin{aligned} w_1 = {} & 3\alpha_1\left[x\left(1 - \mathrm{erf}(x)\right) - \pi^{-1/2}\mathrm{e}^{-x^2}\right]\sin(\theta) + \frac{3}{4}\pi^{-3/2}\left\{\left[2x\pi^{1/2}(3\pi - 2) - 3\pi\left(1 + 2x^2\right)\mathrm{e}^{-x^2}\right]\right. \\ & \left.\times\,\mathrm{erf}(x) + \left[2(3\pi - 2)\mathrm{e}^{-x^2} - 6x\pi^{3/2} + 2x\pi^{1/2}\left(2 - 3\mathrm{e}^{-2x^2} + 4\mathrm{e}^{-x^2}\right)\right]\right\}\sin(2\theta), \end{aligned} \tag{18}$$

and the stream function corresponding to (18) is found by integrating $\frac{\partial^2\Psi_1}{\partial x^2} = -\sin\theta\,\omega_1$ twice subject to $\Psi_1 = \frac{\partial\Psi_1}{\partial x} = 0$ at $x = 0$ and is given by

$$\begin{aligned}\Psi_1 = \frac{1}{4}\,\pi^{-3/2}\Big\{&\alpha_1\left[\pi^{3/2}x\left(3+2x^2\right)\mathrm{erf}(x) - 2\pi\left(1 + x^3\pi^{1/2} - \left(x^2+1\right)\mathrm{e}^{-x^2}\right)\right]\sin^2(\theta)\\&+\Big[9x\pi^{3/2}\,\mathrm{erf}^2(x) + \left(12\pi + 6x\pi^{1/2} - 9x\pi^{3/2} - 6\pi^{3/2}x^3 + 27\pi\mathrm{e}^{-x^2} + 4x^3\pi^{1/2}\right)\mathrm{erf}(x)\\&+\left(4-6\pi\right)\left(x^2+1\right)\mathrm{e}^{-x^2}\\&-2x\pi^{1/2}\left(3+2x^2-3\pi x^2\right) - 18\pi 2^{1/2}\,\mathrm{erf}\left(2^{1/2}x\right) + 6\pi - 4\Big]\sin^2(\theta)\cos(\theta)\Big\}.\end{aligned} \tag{19}$$

Finally, the function ω_2 satisfies the equation

$$\frac{\partial^2\omega_2}{\partial x^2} + 2x\,\frac{\partial\omega_2}{\partial x} - 6\omega_2 = r_1(x)\sin\theta + r_2(x)\sin 2\theta + r_3(x)\sin 3\theta, \tag{20}$$

where $r_1(x)$, $r_2(x)$ and $r_3(x)$ are given by

$$\begin{aligned}r_1 = \frac{9}{4}\pi^{-5/2}\Big\{&\frac{2}{3}\pi\Big[36\pi^{1/2}\left(2x^2-3\right)x\mathrm{e}^{-2x^2} + \Big(\pi\left(6x^4+9x^2+45\right) + \pi^{1/2}\left(66-84x^2\right)x - 4x^4\\&-6x^2-30\Big)\mathrm{e}^{-x^2} - 45\pi^{3/2}x - 18\pi + 12 + 30\pi^{1/2}x\Big]\mathrm{erf}(x) + \pi\Big(10\pi^{1/2}(3\pi-2)x + \pi\\&\times\left(24x^4-36x^2-9\right)\mathrm{e}^{-x^2}\Big)\mathrm{erf}^2(x) - 24\pi\left(1-x^2\right)\mathrm{e}^{-3x^2} + \frac{4}{3}\Big(\pi^{1/2}(3\pi-2)\left(x^2+1\right)x\\&+2\pi\left(15-21x^2\right)\Big)\mathrm{e}^{-2x^2} - \frac{4}{3}\Big(3\pi^2\left(x^4+3\right) + 3\pi^{3/2}x\left[1-3\,(2)^{1/2}\mathrm{erf}\left(2^{1/2}x\right)\right] - 2\pi^{1/2}x\\&-\pi\left(2x^4+27x^2-6\right)\Big)\mathrm{e}^{-x^2} + 4\pi(3\pi-2)\Big\},\end{aligned} \tag{21}$$

$$\begin{aligned}r_2(x) = \pi^{-1/2}\Big[&6\,\alpha_1\left(2x^4+3x^2-3\right)\mathrm{e}^{-x^2} + 18\,\alpha_1\left(x+\pi^{-1/2}\right)\Big]\mathrm{erf}(x) - 18\,\alpha_1\,x\,\mathrm{erf}^2(x)\\&+12\,\alpha_1\pi^{-1}\left(x^2+1\right)x\,\mathrm{e}^{-2x^2} + 6\,\alpha_1\pi^{-1}\left[\left(\left(3-2x^4\right)\pi^{1/2} - 2x\right)\mathrm{e}^{-x^2} - 3\pi^{1/2}\right],\end{aligned} \tag{22}$$

$$\begin{aligned}r_3(x) = -\frac{9}{4}\,\pi^{-5/2}\Big\{&\Big[2\pi\left(2(3\pi-2)x^4 + 28\pi^{1/2}x^3 + 3(3\pi-2)x^2 - 22\pi^{1/2}x + 2 - 3\pi\right)\mathrm{e}^{-x^2}\\&-24\pi^{3/2}\times\left(1+2x^2\right)x\mathrm{e}^{-2x^2} + 2\pi\left(\pi^{1/2}(3\pi-2)x + 6\pi - 4\right)\Big]\mathrm{erf}(x)\\&-\pi\left[3\pi\left(8x^4+4x^2+1\right)\mathrm{e}^{-x^2} + 2\pi^{1/2}(3\pi-2)\,x\right]\mathrm{erf}^2(x) - 4\Big[\pi(3\pi-2)\,x^4 + 5\pi\,x^2\\&+\pi^{1/2}(3\pi-2)\,x - 9\,2^{1/2}\pi^{3/2}\,x\times\mathrm{erf}\left(2^{1/2}x\right) - \pi(3\pi+2)\Big]\mathrm{e}^{-x^2} + 4\Big[\pi^{1/2}(3\pi-2)x^3\\&+14\pi x^2 + \pi^{1/2}(3\pi-2)x - 10\pi\Big]\mathrm{e}^{-2x^2} + 24\pi(1-x^2)\mathrm{e}^{-3x^2} - 4\pi(3\pi-2)\Big\}.\end{aligned} \tag{23}$$

The corresponding solutions for ω_2 and Ψ_2 consist of a sum of three terms in $\sin\theta$, $\sin 2\theta$ and $\sin 3\theta$ with coefficients which are functions of x. Exact expressions for these functions have been obtained; because of their long expression we only present

$$\begin{aligned}\lim_{x\to 0}\omega_2 = &-\frac{1}{320}\,\pi^{-5/2}\left[5\pi^2\left(512{\alpha_2}^2 - 387\right) - 6\pi\left(2161 - 486\sqrt{3}\right) + 5888\right]\sin\theta\\&+\frac{1}{80}\,\alpha_1\pi^{-3/2}\left(495\pi - 896\right)\sin 2\theta\\&-\frac{1}{320}\,\pi^{-5/2}\left[435\pi^2 - 18\pi\left(559 - 234\sqrt{3}\right) + 2816\right]\sin 3\theta.\end{aligned} \tag{24}$$

Thus the approximate solution is given by $\zeta \sim R^{1/2}/(2(2\tau)^{1/2})\left(\omega_0 + \tau\,\omega_1 + \tau^2\,\omega_2\right)$ for large enough R and small τ. The corresponding surface vorticity is

$$\begin{aligned}\zeta(0,\theta,\tau) \sim -\frac{R^{1/2}}{2(2\tau)^{1/2}}\Bigg\{ & \frac{1}{320}\,\pi^{-5/2}\bigg[960\pi^2\,(1+\alpha_1\tau) + \Big(5\pi^2\,(387+512\alpha_2) \\ & - 2\left(\pi\left(6483 - 1458\,3^{1/2}\right) - 2944\right)\Big)\tau^2\bigg]\sin\theta - \frac{1}{80}\,\pi^{-3/2}\,[45\pi(8+11\alpha_1\tau) \\ & - 16(15+56\alpha_1\tau)]\sin 2\theta + \frac{1}{320}\,\pi^{-5/2}\bigg[435\pi^2 - 18\pi\left(559 - 234\,3^{1/2}\right) \\ & + 2816\bigg]\tau^2 \sin 3\theta\Bigg\}.\end{aligned} \tag{25}$$

CALCULATED RESULTS

One of the interesting physical features of the flow is the determination of the time at which the fluid first starts to separate from the sphere. Separation first occurs at $\tau = T$, say, defined by the condition $\frac{\partial\zeta}{\partial\theta} = 0$ for $x = 0$, $\zeta = 0$. From the expansion (25) in powers of τ we can obtain the time T as the positive root of

$$\begin{aligned}-\frac{1}{40}\,\pi^{-5/2}\left[5\pi^2\,(81 - 99\alpha_1 + 64\alpha_2) + \pi\left(6\left(324\,3^{1/2} - 899\right) + 896\alpha_1 + 1792\right)\right]T^2 \\ - 3\pi^{-3/2}\,(2+\pi(\alpha_1 - 3))\,T - 3\pi^{-1/2} = 0.\end{aligned} \tag{26}$$

This yields the value $T = 0.39145$ when $\alpha_1 = \alpha_2 = 0$ which is in good agreement with the approximation given by Boltze [1] for the uniform transient flow over a sphere. Boltze gives T for the boundary-layer case as 0.392. His series was also computed up to τ^3. It is also noted that Dennis and Walker [3] give T for the case of impulsively started sphere without acceleration as 0.396. In their investigation, the series was computed up to τ^7, numerically, and this produced a greater value for T than our result did. Approximations corresponding to the values of α_2 have been calculated when $\alpha_1 = 0, 0.05$ and these are shown in Tables 1 and 2, respectively. These results show the effect of increase of α_2 is to increase the time of separation. T is also determined for the oscillating sphere case when $\alpha_1 = 0$ and $\alpha_2 = -1/2\alpha^2$ for selected values of α and the results are given in Table 3. This table shows that the effect of increase α is to drastically reduce the time of separation. It is noted that Smith [5,6] shows, in the Stokes limit, separation occurs when the sphere is decelerating but the vortex disappears when flow reversal occurs. Results of Table 3 confirm that separation first occurs during the first quarter cycle of the oscillation when the sphere is decelerating. However, it is not possible to produce the streamline patterns to investigate the details of vortex formation in this study. This is due to the presence of an integral, in the streamfunction component Ψ_2 which cannot be evaluated analytically. This will be the subject of a further investigation in which the solution will be obtained numerically to extend the results of the present study for large values of the time.

Table 1. Approximations to the time of separation, T, for the accelerated flow case when $\alpha_1 = 0$.

α_2	0	0.01	0.05	0.1	0.25	0.5
T	0.39145	0.39295	0.39918	0.407538	0.437561	0.437561

Table 2. Approximations to the time of separation, T, for the accelerated flow case when $\alpha_1 = 0.05$.

α_2	0	0.01	0.05	0.1	0.25	0.5
T	0.39368	0.39519	0.40144	0.40982	0.43986	0.52122

Table 3. Approximations to the time of separation, T, for the oscillating flow case: $\alpha_1 = 0$, $\alpha_2 = -(1/2)\alpha^2$.

α	0	$\pi/4$	$\pi/2$	π	2π	4π
T	0.39145	0.35372	0.28931	0.19761	0.11675	0.06346

CONCLUSION

Initial boundary-layer flow of an oscillating fluid in the presence of a sphere is obtained analytically. We may note that unless a boundary-layer structure is adopted there is a considerable difficulty in determining the initial flow by numerical methods due to a singularity in the vorticity at $\tau = 0$. In other words, numerical methods which calculate initial flow in the natural physical coordinates are likely to give inaccurate initial results. The analytical expressions are satisfactory at small times and are useful in providing initial details of the flow. These details can be used as starting information to which the numerical methods may be linked.

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